

Marcinkiewicz's strong law of large numbers for non-additive expectation

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Abstract

The sub-linear expectation space is a nonlinear expectation space having advantages of modelling the uncertainty of probability and distribution. In the sub-linear expectation space, we use capacity and sub-linear expectation to replace probability and expectation of classical probability theory. In this paper, the method of selecting subsequence is used to prove Marcinkiewicz type strong law of large numbers under sub-linear expectation space. This result is a natural extension of the classical Marcinkiewicz's strong law of large numbers to the case where the expectation is nonadditive. In addition, this paper also gives a theorem about convergence of a random series.

keywords: strong law of large numbers, capacity, nonlinear expectation

1 Introduction

Under the frame of classical probability theory established by Kolmogorov, probability and expectation are both linear. To model uncertain phenomena in many areas, like finance, economics and insurance, sublinear expectation is introduced by Peng [2]. We use the notion of independence and identical distribution introduced by Peng. The main purpose of this paper is to establish Marcinkiewicz's strong law of large numbers under nonlinear expectation. Chen [1] obtained Kolmogorov's strong law of large numbers for i.i.d. random variables under the condition of $1 + \alpha$ -moments. Zhang [10] obtained the same result when the corresponding Choquet integral is finite. This can be regarded as a generalization of Kolmogorov's strong law of large numbers.

Our paper is organized as follows: we introduce some basic settings and notations in section 2. In section 3, under the frame introduced by Peng [2], we establish Marcinkiewicz's strong law of large numbers. Then, we introduce a theorem about convergence of random series.

2 Basic Setting

We use notations of Peng [2]. Given a positive integer number n , we denote by $\langle \mathbf{x}, \mathbf{y} \rangle$ the scalar product of $\mathbf{x}, \mathbf{y} \in R^n$ and by $|\mathbf{x}| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$ the Euclidean norm of \mathbf{x} .

Let (Ω, \mathcal{F}) be a measurable space and \mathcal{H} be a linear space of real functions defined on (Ω, \mathcal{F}) such that $\varphi(X_1, X_2, \dots, X_n) \in \mathcal{H}$, for any $X_1, X_2, \dots, X_n \in \mathcal{H}$, $\varphi \in C_{l,Lip}(R^n)$. where $C_{l,Lip}(R^n)$ denotes the linear space of local Lipschitz functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \quad \forall \mathbf{x}, \mathbf{y} \in R^n,$$

for some $C > 0$, $m \in N$ depending on φ .

\mathcal{H} is considered as a space of "random variables".

Definition 1. $\hat{E} : \mathcal{H} \rightarrow (-\infty, +\infty)$ is called sublinear expectation, if \hat{E} satisfies the following properties: $\forall X, Y \in \mathcal{H}$, we have

1. *Monotonicity:* If $X \geq Y$, then $\hat{E}(X) \geq \hat{E}(Y)$.
2. *Constant preserving:* $\hat{E}(c) = c$.
3. *Sub-additivity:* $\hat{E}(X + Y) \leq \hat{E}(X) + \hat{E}(Y)$.
4. *Positive homogeneity:* $\hat{E}(\lambda X) = \lambda \hat{E}(X)$, $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sub-linear expectation space, compared with the classical probability space (Ω, \mathcal{F}, P) . For convenience, we also call \hat{E} a nonlinear expectation.

In a sub-linear expectation space, we replace the concept of probability with the concept of capacity.

Definition 2. A set function $V : \mathcal{F} \rightarrow [0, 1]$ is called a capacity, if

1. $V(\emptyset) = 0, V(\Omega) = 1$.
2. $V(A) \leq V(B), \forall A \subset B, A, B \in \mathcal{F}$.

In addition, if V is continuous, then V should also satisfy

3. $V(A_n) \uparrow V(A), \text{ if } A_n \uparrow A$.

4. $V(A_n) \downarrow V(A)$, if $A_n \downarrow A$, where $A_n, A \in \mathcal{F}$.

A sub-linear expectation \hat{E} could generate a pair of capacity denoted by

$$\hat{V}(A) = \hat{E}(I_A), \hat{v}(A) = -\hat{E}(-I_A), \text{ if } I_A \in \mathcal{H}.$$

We also have the fact

$$\hat{E}(f) \leq V(A) \leq \hat{E}(g), \hat{e}(f) \leq v(A) \leq \hat{e}(g), \text{ if } f \leq I_A \leq g, f, g \in \mathcal{H}$$

It is easy to check that $\hat{V}(A) + \hat{v}(A) = 1$.

We define the Choquet integrals/expectations by

$$C_V(X) = \int_0^\infty V(X \geq t)dt + \int_{-\infty}^0 (V(X \geq t) - 1)dt.$$

where V can also be replaced by v .

Under some conditions, the Borel-Cantelli lemma is still true for capacity.

Lemma 1. [1] Let $\{A_n, n \geq 1\}$ be a sequence of events in \mathcal{F} . If V is continuous and $\sum_{n=1}^\infty V(A_n) < +\infty$, then $V\left(\bigcap_{n=1}^\infty \bigcup_{i=n}^\infty A_i\right) = 0$.

Proof. By the monotonicity and continuity, it follows that

$$0 \leq V\left(\bigcap_{n=1}^\infty \bigcup_{i=n}^\infty A_i\right) \leq V\left(\bigcup_{i=n}^\infty A_i\right) \leq \sum_{i=n}^\infty V(A_i) \rightarrow 0, \text{ if } n \rightarrow \infty. \quad \square$$

Definition 3. [2](Independence) In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, for each test function $\varphi \in C_{l,Lip}(R^m \times R^n)$, $\mathbf{Y} = (Y_1, \dots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to $\mathbf{X} = (X_1, \dots, X_m)$, $X_i \in \mathcal{H}$, if we have $\hat{E}(\varphi(\mathbf{X}, \mathbf{Y})) = \hat{E}\left(\hat{E}(\varphi(\mathbf{x}, \mathbf{Y}))\Big|_{\mathbf{x}=\mathbf{X}}\right)$, whenever $\bar{\varphi}(\mathbf{x}) := \hat{E}[|\varphi(\mathbf{x}, \mathbf{Y})|] < \infty$ for all \mathbf{x} and $\hat{E}[|\varphi \mathbf{X}|] < \infty$.

Definition 4. [2](Identical distribution) Let \mathbf{X}_1 and \mathbf{X}_2 be two n -dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. \mathbf{X}_1 and \mathbf{X}_2 are called identically distributed, denoted by $\mathbf{X}_1 \stackrel{d}{=} \mathbf{X}_2$, if

$$\hat{E}_1(\varphi(\mathbf{X}_1)) = \hat{E}_2(\varphi(\mathbf{X}_2)), \forall \varphi \in C_{l,Lip}(R^n),$$

whenever the sub-expectation is finite.

Definition 5. [2](IID random variables) A sequence of random variables $\{X_n; n \geq 1\}$ is said to be independent, if X_{i+1} is independent to (X_1, \dots, X_i) for each $i \geq 1$. It is said to be identically distributed, if $X_i \stackrel{d}{=} X_1$ for each $i \geq 1$.

3 Main results

In classical probability theory, the proof of Marcinkiewicz's strong law of large numbers depends on the three series theorem. In this paper, we use the method of picking up subsequence to prove it. First, we prove a lemma.

Lemma 2. *Let $C_V(|X|^p) < \infty, 1 \leq p < 2$. X is a random variable defined on sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$. Then*

$$\sum_{j=1}^{\infty} \frac{\hat{E}(|X| \wedge j^{1/p})^2}{j^{2/p}} < \infty.$$

Proof. By noting that

$$\begin{aligned} (|X| \wedge j^{1/p})^2 &= \sum_{i=1}^j |X|^2 I\left((i-1)^{1/p} < |X| \leq i^{1/p}\right) + j^{2/p} I(|X| > j^{1/p}) \\ &\leq \sum_{i=1}^j i^{2/p} I\left((i-1)^{1/p} < |X| \leq i^{1/p}\right) + j^{2/p} I(|X| > j^{1/p}) \\ &= \sum_{i=0}^{j-1} (i+1)^{2/p} I(|X| > i^{1/p}) - \sum_{i=1}^j i^{2/p} I(|X| > i^{1/p}) + j^{2/p} I(|X| > j^{1/p}) \\ &= 1 + \sum_{i=1}^{j-1} \left((i+1)^{2/p} - i^{2/p}\right) I(|X| > i^{1/p}) \\ &\leq 1 + \sum_{i=1}^j \left((i+1)^{2/p} - i^{2/p}\right) I(|X| > i^{1/p}), \end{aligned}$$

we have,

$$\hat{E}\left((X \wedge j^{1/p})^2\right) \leq 1 + \sum_{i=1}^j \left((i+1)^{2/p} - i^{2/p}\right) V(|X| > i^{1/p}).$$

Hence,

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{\hat{E}\left((X \wedge j^{1/p})^2\right)}{j^{2/p}} &\leq \sum_{j=1}^{\infty} \frac{1}{j^{2/p}} + \sum_{j=1}^{\infty} j^{-2/p} \sum_{i=1}^j \left((i+1)^{2/p} - i^{2/p}\right) V(|X| > i^{1/p}) \\ &\leq \sum_{j=1}^{\infty} \frac{1}{j^{2/p}} + \sum_{i=1}^{\infty} \left((i+1)^{2/p} - i^{2/p}\right) V(|X| > i^{1/p}) \sum_{j=i}^{\infty} j^{-2/p} \\ &\leq \sum_{j=1}^{\infty} \frac{1}{j^{2/p}} + C \sum_{i=1}^{\infty} \frac{\left((i+1)^{2/p} - i^{2/p}\right)}{i^{2/p-1}} V(|X| > i^{1/p}), \end{aligned}$$

where C is a positive constant.

Let $f(x) = \frac{(x+1)^{2/p} - x^{2/p}}{x^{2/p} - 1} = x(1 + \frac{1}{x})^{2/p} - x, x \geq 1$.

Since $f'(x) = (1 + \frac{1}{x})^{2/p-1} - 1 > 0$, $f(x)$ is increasing. And $\lim_{x \rightarrow \infty} f(x) = \frac{2}{p}$, so,

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{\hat{E}\left((X \wedge j^{1/p})^2\right)}{j^{2/p}} &\leq \sum_{j=1}^{\infty} \frac{1}{j^{2/p}} + C \sum_{i=1}^{\infty} \frac{\left((i+1)^{2/p} - i^{2/p}\right)}{i^{2/p} - 1} V(|X| > i^{1/p}) \\ &\leq \sum_{j=1}^{\infty} \frac{1}{j^{2/p}} + C \cdot \frac{2}{p} \sum_{i=1}^{\infty} V(|X| > i^{1/p}) \\ &\leq \sum_{j=1}^{\infty} \frac{1}{j^{2/p}} + C \cdot \frac{2}{p} \cdot C_V(|X|^p) < \infty. \end{aligned}$$

□

Theorem 6. [10] Suppose that X_1, \dots, X_n is a sequence of independent random variable and $\hat{E}(X_k) \leq 0, k = 1, \dots, n$. Then

$$\hat{E}\left(\left|\max_{k \leq n} (S_n - S_k)\right|^p\right) \leq 2^{2-p} \sum_{k=1}^n \hat{E}(|X_k|^p), 1 \leq p \leq 2,$$

where $S_k = X_1 + \dots + X_k$.

In particular, $\hat{E}((S_n^+)^p) \leq 2^{2-p} \sum_{k=1}^n \hat{E}(|X_k|^p), 1 \leq p \leq 2$.

By Lemma 2, we can prove Marcinkiewz's strong law of large numbers.

Theorem 7. In sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, $\{X_i\}$ is a sequence of independent and identical random variable. Suppose that $1 \leq p < 2$, $\hat{E}(X_i) = \hat{\varepsilon}(X_i) = 0$, $\lim_{c \rightarrow \infty} \hat{E}[(|X_1|^p - c)^+] = 0$ and V is continuous, then $C_V(|X_1|^p) < \infty$ if and only if $S_n/n^{1/p} \rightarrow 0$ a.s. V .

Proof. \Rightarrow Define $Y_k = (-k^{1/p}) \vee (X_k \wedge k^{1/p})$.

So,

$$\sum_{k=1}^{\infty} V(Y_k \neq X_k) = \sum_{k=1}^{\infty} V(|X_1| > k^{1/p}) \leq \int_0^{\infty} V(|X_1| > t^{1/p}) dt = C_V(|X_1|^p) < \infty.$$

By the Borel-Cantelli lemma, we can get $V(X_k \neq Y_k, i.o.) = 0$. That is

$$v\left(\liminf_{k \rightarrow \infty} (X_k = Y_k)\right) = 1.$$

$\forall \omega \in \liminf_{k \rightarrow \infty} (X_k = Y_k), \exists K$, s.t.

$$X_k(\omega) = Y_k(\omega), \text{ when } k > K.$$

so $\sum_{k \geq K} X_k = \sum_{k \geq K} Y_k$.

In order to prove $S_n/n^{1/p} \rightarrow 0$ a.s. V , we only need to prove $\bar{S}_n/n^{1/p} \rightarrow 0$ a.s. V , where $\bar{S}_n = \sum_{k=1}^n Y_k$. Let $n_k = [\theta]^k$, $\theta > 1$.

Step one: we first prove $\frac{\bar{S}_{n_k}}{n_k^{1/p}} \rightarrow 0$ a.s. V .

$$V \left(\frac{\bar{S}_{n_k}}{n_k^{1/p}} \geq \varepsilon \right) = V \left(\sum_{j=1}^{n_k} (Y_j - \hat{E}Y_j) \geq \varepsilon \cdot n_k^{1/p} - \sum_{j=1}^{n_k} \hat{E}Y_j \right).$$

Now we prove $\frac{1}{2}\varepsilon \cdot n_k^{1/p} - \sum_{j=1}^{n_k} \hat{E}Y_j \geq 0$, i.e. $\frac{2 \sum_{j=1}^{n_k} \hat{E}Y_j}{(n_k)^{1/p}} \leq \varepsilon$.

Since $\hat{E}(X_j) = 0$, we have

$$\begin{aligned} |\hat{E}Y_j| &= \left| \hat{E}(X_j I(|X_j| \leq j^{1/p}) + j^{1/p} I(|X_j| > j^{1/p})) \right| \\ &\leq \left| \hat{E}(X_j I(|X_j| \leq j^{1/p})) \right| + \left| \hat{E}(j^{1/p} I(|X_j| > j^{1/p})) \right| \\ &\leq \left| \hat{E}(X_j (1 - I(|X_j| > j^{1/p}))) \right| + \left| \hat{E}(|X_j| I(|X_j| > j^{1/p})) \right| \\ &\leq 2 \cdot \left| \hat{E}(|X_j| I(|X_j| > j^{1/p})) \right| = 2j^{1/p} \cdot \left| \hat{E}\left(\frac{|X_j|}{j^{1/p}} I(|X_j| > j^{1/p})\right) \right| \\ &\leq 2j^{(1/p-1)} \cdot \left| \hat{E}(|X_1|^p I(|X_1|^p > j)) \right| \end{aligned}$$

Hence,

$$\sum_{j=1}^{n_k} |\hat{E}Y_j| \leq \sum_{j=1}^{n_k} 2j^{(1/p-1)} \hat{E}(|X_1|^p I(|X_1|^p > j)).$$

Since $\lim_{c \rightarrow \infty} \hat{E}[(|X_1|^p - c)^+] = 0$, then

$$\hat{E}(|X_1|^p I(|X_1|^p > j)) \rightarrow 0, \text{ when } j \rightarrow \infty.$$

Therefore,

$$\frac{\sum_{j=1}^{n_k} j^{(1/p-1)} \hat{E}(|X_1|^p I(|X_1|^p > j))}{(n_k)^{1/p}} \rightarrow 0, \text{ when } k \rightarrow \infty.$$

We can get $\frac{1}{2}\varepsilon \cdot n_k^{1/p} - \sum_{j=1}^{n_k} |\hat{E}Y_j| \geq 0$. Furthermore, $\frac{1}{2}\varepsilon \cdot n_k^{1/p} - \sum_{j=1}^{n_k} \hat{E}Y_j \geq 0$.

It's obvious that $\hat{E}(Y_j - \hat{E}Y_j) \leq 0$.

By theorem 5,

$$\begin{aligned}
V\left(\frac{\bar{S}_{n_k}}{n_k^{1/p}} \geq \varepsilon\right) &= V\left(\sum_{j=1}^{n_k} (Y_j - \hat{E}Y_j) \geq \varepsilon \cdot n_k^{1/p} - \sum_{j=1}^{n_k} \hat{E}Y_j\right) \\
&\leq V\left(\sum_{j=1}^{n_k} (Y_j - \hat{E}Y_j) \geq \frac{1}{2}\varepsilon \cdot n_k^{1/p}\right) \leq V\left(\sum_{j=1}^{n_k} (Y_j - \hat{E}Y_j)^+ \geq \frac{1}{2}\varepsilon \cdot n_k^{1/p}\right) \\
&\leq \frac{\hat{E}\left(\left(\sum_{j=1}^{n_k} (Y_j - \hat{E}Y_j)^+\right)^2\right)}{\left(\frac{1}{2}\varepsilon \cdot n_k^{1/p}\right)^2} \leq \frac{\hat{E}\left(\left|\max_{1 \leq j \leq n_k} \sum_{i=1}^j (Y_i - \hat{E}Y_i)^+\right|^2\right)}{\left(\frac{1}{2}\varepsilon \cdot n_k^{1/p}\right)^2} \\
&\leq \frac{\sum_{j=1}^{n_k} \hat{E}\left(|Y_j - \hat{E}Y_j|^2\right)}{\left(\frac{1}{2}\varepsilon \cdot n_k^{1/p}\right)^2} \leq \frac{4 \sum_{j=1}^{n_k} \hat{E}\left(|Y_j|^2\right)}{\left(\frac{1}{2}\varepsilon \cdot n_k^{1/p}\right)^2}.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{n_k^{2/p}} \cdot \sum_{j=1}^{n_k} \hat{E}\left(|Y_j|^2\right) &= \sum_{j=1}^{\infty} \hat{E}\left(|Y_j|^2\right) \cdot \sum_{k:n_k \geq j} \frac{1}{n_k^{2/p}} \\
&\leq \sum_{j=1}^{\infty} \frac{\hat{E}\left(|Y_j|^2\right)}{j^{2/p}} \leq \sum_{j=1}^{\infty} \frac{\hat{E}\left(|X_j \wedge j^{1/p}|^2\right)}{j^{2/p}} < \infty,
\end{aligned}$$

we have

$$\sum_{k=1}^{\infty} V\left(\frac{\bar{S}_{n_k}}{n_k^{1/p}} \geq \varepsilon\right) < \infty.$$

By the Borel-Cantelli lemma, we can get $V\left(\frac{\bar{S}_{n_k}}{n_k^{1/p}} \geq \varepsilon, i.o.\right) = 0$.

Considering $\{-X_i\}$, by the same way, we can get

$$V\left(\frac{-\bar{S}_{n_k}}{n_k^{1/p}} \geq \varepsilon, i.o.\right) = 0.$$

By noting that,

$$V\left(\frac{|\bar{S}_{n_k}|}{n_k^{1/p}} \geq \varepsilon, i.o.\right) \leq V\left(\frac{\bar{S}_{n_k}}{n_k^{1/p}} \geq \varepsilon, i.o.\right) + V\left(\frac{-\bar{S}_{n_k}}{n_k^{1/p}} \geq \varepsilon, i.o.\right) = 0.$$

Hence,

$$\frac{\bar{S}_{n_k}}{n_k^{1/p}} \rightarrow 0 \quad a.s. \ V.$$

Step two: Next, we will prove $\frac{\max_{n_{k-1} < n \leq n_k} |\bar{S}_n - \bar{S}_{n_k}|}{n_k^{1/p}} \rightarrow 0 \text{ a.s. } V$.

We only prove $\frac{\max_{n_{k-1} < n \leq n_k} (\bar{S}_{n_k} - \bar{S}_n)^+}{n_k^{1/p}} \rightarrow 0 \text{ a.s. } V$, because the proof of $\frac{\max_{n_{k-1} < n \leq n_k} (\bar{S}_{n_k} - \bar{S}_n)^-}{n_k^{1/p}} \rightarrow 0 \text{ a.s. } V$ is similar.

$$\begin{aligned} & V \left(\max_{n_{k-1} < n \leq n_k} (\bar{S}_{n_k} - \bar{S}_n)^+ \geq \varepsilon \cdot n_k^{1/p} \right) \\ & \leq V \left(\max_{n_{k-1} < n \leq n_k} \left(\bar{S}_{n_k} - \bar{S}_n - \sum_{j=n+1}^{n_k} \hat{E}Y_j \right)^+ \geq \varepsilon \cdot n_k^{1/p} - \sum_{j=n_{k-1}+1}^{n_k} |\hat{E}Y_j| \right). (*) \end{aligned}$$

Now, we prove $\frac{1}{2}\varepsilon \cdot n_k^{1/p} - \sum_{j=n_{k-1}+1}^{n_k} |\hat{E}Y_j| \geq 0$, i.e. $\frac{2 \sum_{j=n_{k-1}+1}^{n_k} |\hat{E}Y_j|}{(n_k)^{1/p}} \leq \varepsilon$.

Since $\hat{E}(X_j) = 0$, we can get

$$\begin{aligned} |\hat{E}Y_j| &= \left| \hat{E}(X_j I(|X_j| \leq j^{1/p}) + j^{1/p} I(|X_j| > j^{1/p})) \right| \\ &\leq \left| \hat{E}(X_j I(|X_j| \leq j^{1/p})) \right| + \left| \hat{E}(j^{1/p} I(|X_j| > j^{1/p})) \right| \\ &\leq \left| \hat{E}(X_j (1 - I(|X_j| > j^{1/p}))) \right| + \left| \hat{E}(|X_j| I(|X_j| > j^{1/p})) \right| \\ &\leq 2 \cdot \left| \hat{E}(|X_j| I(|X_j| > j^{1/p})) \right| = 2j^{1/p} \cdot \left| \hat{E}\left(\frac{|X_j|}{j^{1/p}} I(|X_j| > j^{1/p})\right) \right| \\ &\leq 2j^{(1/p-1)} \cdot \left| \hat{E}(|X_1|^p I(|X_1|^p > j)) \right| \end{aligned}$$

So,

$$\sum_{j=n_{k-1}+1}^{n_k} |\hat{E}Y_j| \leq \sum_{j=n_{k-1}+1}^{n_k} 2j^{(1/p-1)} \hat{E}(|X_1|^p I(|X_1|^p > j)).$$

Since $\lim_{c \rightarrow \infty} \hat{E}(|X_1|^p - c)^+ = 0$, then

$$\hat{E}(|X_1|^p I(|X_1|^p > j)) \rightarrow 0, \text{ when } j \rightarrow \infty.$$

Therefore,

$$\frac{\sum_{j=n_{k-1}+1}^{n_k} j^{(1/p-1)} \hat{E}(|X_1|^p I(|X_1|^p > j))}{(n_k)^{1/p}} \rightarrow 0, \text{ when } k \rightarrow \infty.$$

We can get $\frac{1}{2}\varepsilon \cdot n_k^{1/p} - \sum_{j=n_{k-1}+1}^{n_k} |\hat{E}Y_j| \geq 0$.

since $\hat{E} \left(Y_i - \hat{E}(Y_i) \right) \leq 0$, (*) can be transformed into

$$\begin{aligned}
& V \left(\max_{n_{k-1} < n \leq n_k} (\bar{S}_{n_k} - \bar{S}_n)^+ \geq \varepsilon \cdot n_k^{1/p} \right) \\
& \leq V \left(\max_{n_{k-1} < n \leq n_k} \left(\bar{S}_{n_k} - \bar{S}_n - \sum_{j=n+1}^{n_k} \hat{E} Y_j \right)^+ \geq \frac{1}{2} \varepsilon n_k^{1/p} \right) \\
& = V \left(\max_{n_{k-1} < n \leq n_k} \left(\sum_{j=n+1}^{n_k} (Y_j - \hat{E} Y_j) \right)^+ \geq \frac{1}{2} \varepsilon n_k^{1/p} \right) \\
& \leq \frac{\hat{E} \left(\left(\max_{n_{k-1} < n \leq n_k} \left(\sum_{j=n+1}^{n_k} (Y_j - \hat{E} Y_j) \right)^+ \right)^2 \right)}{\left(\frac{1}{2} \varepsilon \cdot n_k^{1/p} \right)^2} \\
& \leq \frac{\sum_{j=n_{k-1}+1}^{n_k} \hat{E} \left[|Y_j - \hat{E} Y_j|^2 \right]}{\left(\frac{1}{2} \varepsilon \cdot n_k^{1/p} \right)^2} \leq \frac{4 \sum_{j=n_{k-1}+1}^{n_k} \hat{E} [|Y_j|^2]}{\left(\frac{1}{2} \varepsilon \cdot n_k^{1/p} \right)^2}.
\end{aligned}$$

So,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\sum_{j=n_{k-1}+1}^{n_k} \hat{E} [|Y_j|^2]}{n_k^{2/p}} & \leq \sum_{k=1}^{\infty} \sum_{j=n_{k-1}+1}^{n_k} \frac{\hat{E} [|Y_j|^2]}{j^{2/p}} = \sum_{j=2}^{\infty} \frac{\hat{E} [|Y_j|^2]}{j^{2/p}} \\
& < \sum_{j=1}^{\infty} \frac{\hat{E} [|X_j \wedge j^{1/p}|^2]}{j^{2/p}} < \infty.
\end{aligned}$$

By Borel-Cantelli lemma, we can get

$$V \left(\frac{\max_{n_{k-1} < n \leq n_k} (\bar{S}_{n_k} - \bar{S}_n)^+}{n_k^{1/p}} \geq \varepsilon, i.o. \right) = 0.$$

Hence,

$$\frac{\max_{n_{k-1} < n \leq n_k} (\bar{S}_{n_k} - \bar{S}_n)^+}{n_k^{1/p}} \rightarrow 0 \text{ a.s. } V.$$

Noting that $|\bar{S}_{n_k} - \bar{S}_n| = (\bar{S}_{n_k} - \bar{S}_n)^+ + (\bar{S}_{n_k} - \bar{S}_n)^-$, finally we can get,

$$\frac{\max_{n_{k-1} < n \leq n_k} |\bar{S}_n - \bar{S}_{n_k}|}{n_k^{1/p}} \rightarrow 0 \text{ a.s. } V.$$

\Leftarrow Suppose $C_V(|X_1|^p) = \infty$.

Let g_ε be a function satisfying that its derivatives of each order are bounded, $g_\varepsilon(x) = 1$ if $x > 1$, $g_\varepsilon(x) = 0$ if $x \leq 1 - \varepsilon$, and $0 \leq g_\varepsilon(x) \leq 1$ for all x , where $0 < \varepsilon < 1$. Then

$$g_\varepsilon(\cdot) \in C_{l,Lip}(R) \text{ and } I\{x \geq 1\} \leq g_\varepsilon(x) \leq I\{x > 1 - \varepsilon\}.$$

So,

$$\begin{aligned} \sum_{j=1}^{\infty} \hat{E} \left(g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) \right) &= \sum_{j=1}^{\infty} \hat{E} \left(g_{1/2} \left(\frac{|X_1|}{(Mj)^{1/p}} \right) \right) \\ &\geq \sum_{j=1}^{\infty} V(|X_1| > (Mj)^{1/p}) = \infty, \quad \forall M > 0. (**) \end{aligned}$$

For any $l \geq 1$,

$$\begin{aligned} v \left(\sum_{j=1}^n g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) < l \right) &= v \left(\exp \left\{ -\frac{1}{2} \sum_{j=1}^n g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) \right\} > \exp \left(-\frac{l}{2} \right) \right) \\ &\leq e^{l/2} \hat{e} \left(\exp \left\{ -\frac{1}{2} \sum_{j=1}^n g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) \right\} \right) \\ &\leq e^{l/2} \prod_{j=1}^n \hat{e} \left(\exp \left\{ -\frac{1}{2} g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) \right\} \right) \end{aligned}$$

by the independence and $0 \leq \exp \left\{ -\frac{1}{2} \sum_{j=1}^n g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) \right\} \in C_{l,Lip}(R)$.

Applying the elementary inequality

$$e^{-x} \leq 1 - \frac{1}{2}x \leq e^{-x/2}, \quad \forall 0 \leq x \leq \frac{1}{2}$$

yields

$$\exp \left\{ -\frac{1}{2} g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) \right\} \leq 1 - \frac{1}{4} g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) \leq \exp \left\{ -\frac{1}{4} g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) \right\}.$$

It follows that

$$\hat{e} \left(\exp \left\{ -\frac{1}{2} g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) \right\} \right) \leq \hat{e} \left(\exp \left\{ -\frac{1}{4} g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) \right\} \right).$$

since

$$\begin{aligned} \hat{e} \left(\exp \left\{ -\frac{1}{4} g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) \right\} \right) &\leq \exp \left\{ \hat{e} \left(-\frac{1}{4} g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) \right) \right\} \text{ and} \\ \exp \left\{ \hat{e} \left(-\frac{1}{4} g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) \right) \right\} &= \exp \left\{ -\frac{1}{4} \hat{E} \left(g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) \right) \right\}, \end{aligned}$$

we have

$v \left(\sum_{j=1}^n g_{1/2} \left(\frac{|X_1|}{(Mj)^{1/p}} \right) < l \right) \leq e^{\frac{l}{2}} \exp \left\{ -\frac{1}{4} \hat{E} \left(g_{1/2} \left(\frac{|X_1|}{(Mj)^{1/p}} \right) \right) \right\} \rightarrow 0$, as $n \rightarrow \infty$.
By (**), we have

$$V \left(\sum_{j=1}^n g_{1/2} \left(\frac{|X_1|}{(Mj)^{1/p}} \right) > l \right) \rightarrow 1, \text{ as } n \rightarrow \infty.$$

By continuity of V , we can get

$$\begin{aligned} V \left(\limsup_{n \rightarrow \infty} \frac{|X_n|}{n^{1/p}} > \frac{M}{2} \right) &= V \left(\frac{|X_j|}{(Mj)^{1/p}}, i.o \right) \\ &\geq V \left(\sum_{j=1}^{\infty} g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) = \infty \right) \\ &= \lim_{l \rightarrow \infty} V \left(\sum_{j=1}^{\infty} g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) > \frac{l}{2} \right) \\ &= \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} V \left(\sum_{j=1}^{\infty} g_{1/2} \left(\frac{|X_j|}{(Mj)^{1/p}} \right) > \frac{l}{2} \right) = 1. \end{aligned}$$

On the other hand,

$$\limsup_{n \rightarrow \infty} \frac{|X_n|}{n^{1/p}} \leq \limsup_{n \rightarrow \infty} \left(\frac{|S_n|}{n^{1/p}} + \frac{|S_{n-1}|}{n^{1/p}} \right) \leq 2 \limsup_{n \rightarrow \infty} \frac{|S_n|}{n^{1/p}}.$$

It follows that

$$V \left(\limsup_{n \rightarrow \infty} \frac{|X_n|}{n^{1/p}} > m \right) = 1, \forall m > 0.$$

This contradicts $V \left(\lim_{n \rightarrow \infty} \frac{|X_n|}{n^{1/p}} = 0 \right) = 1$.

Therefore, the assumption $C_V(|X_1|^p) = \infty$ is incorrect.

Finally, we have

$$C_V(|X_1|^p) < \infty.$$

□

Next, we will give a theorem about convergence of a random series

Theorem 8. [10] Suppose that $\{X_i\}$ is a sequence of independent random variable in a sub-linear expectation space. If $\hat{E}(X_i) = \hat{\varepsilon}(X_i) = 0$, $S_k = \sum_{i=1}^k X_i$, then

$$\hat{E} \left(\max_{k \leq n} |S_k|^p \right) \leq C_p \left\{ \sum_{k=1}^n \hat{E} (|X_k|^p) + \left(\sum_{k=1}^n \hat{E} (|X_k|^2) \right)^{p/2} \right\}.$$

where C_p is a positive constant depending only on p .

Definition 9. [5] A sub-linear expectation \hat{E} is called regular, if for any random variable sequence $\{X_n\}$ such that $X_n \downarrow 0$, we have $\hat{E}(X_n) \downarrow 0$.

Theorem 10. [5] If the sub-linear expectation \hat{E} is regular, and $\{\xi_n\}$ is a Cauchy sequence in capacity, there exists a subsequence $\{\xi_{n_k}\}$ converges to some ξ almost surely in capacity.

Theorem 11. Suppose that $\{X_i\}$ is a sequence of independent random variable. If \hat{E} is regular, $\hat{E}(X_i) = \hat{\varepsilon}(X_i) = 0$, and $\sum_{i=1}^{\infty} \hat{E}(X_i^2) < \infty$, then $\sum_{i=1}^{\infty} X_i$ converges almost surely in capacity.

Proof. Define $S_n = \sum_{i=1}^n X_i$. By the independence of random variable, $\forall \varepsilon > 0$, if $m > n$, then

$$V(|S_m - S_n| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} \hat{E}((S_m - S_n)^2) = \frac{1}{\varepsilon^2} \sum_{k=n+1}^m \hat{E}(X_k^2) \rightarrow 0, \text{ when } m \rightarrow \infty.$$

So S_n is a Cauchy sequence in capacity.

Note that \hat{E} is regular, so there exist a subsequence $\{n_k\}$ such that

$$S_{n_k} \rightarrow S \text{ a.s. } V.$$

By Chebyshev inequality and theorem 8 with $p = 2$, we can get

$$\begin{aligned} \sum_{k=1}^{\infty} V \left(\max_{n_k < j \leq n_{k+1}} |S_j - S_{n_k}| \geq \varepsilon \right) &\leq \sum_{k=1}^{\infty} \frac{\hat{E} \left(\max_{n_k < j \leq n_{k+1}} |S_j - S_{n_k}|^2 \right)}{\varepsilon^2} \\ &\leq \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} 2C_2 \sum_{j=n_k+1}^{n_{k+1}} \hat{E}(|X_k|^2) \\ &\leq \frac{2C_2}{\varepsilon^2} \sum_{k=1}^{\infty} \hat{E}(|X_k|^2). \end{aligned}$$

By Borel-Cantelli lemma, we can get

$$\max_{n_k < j \leq n_{k+1}} |S_j - S_{n_k}| \rightarrow 0 \text{ a.s. } V.$$

Hence,

$$S_n \rightarrow S \text{ a.s. } V.$$

□

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